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Translated by M.D.F.

PMM U.S.S.R., Vol.48, No.6, pp. 757-764, 1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

EQUILIBRIUM OF AN ELASTIC LAYER WEAKENED BY PLANE CRACKS*

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The spatial problem of the elastic equilibrium of a layer in whose middle plane there is a system of cracks is considered. The cracks are maintained open under the action of a normal load applied to their edges. The layer faces are compressed between two rigid smooth foundations. The problem is reduced to solving an integral equation of the first kind. The asymptotic methods of "large and small λ" /1/ as well as the method of successive approximations and a variational method are used to construct the solutions of this equation for elliptically and rectangularly shaped cracks in different ranges of variation of the geomtrical parameters.

1. Formulation of the problem, properties of the kernel of the integral equation. Let a domain occupied by an elastic medium be determined by the inequalities $|z| \leqslant h$, $|x| < \infty$, $|y| < \infty$. A crack occupying a certain domain Ω in planform is in the $z = 0$ plane. A load $\sigma_z = -p(x, y)$, $z = \pm 0$ is applied to the crack edges. The following conditions are realized on the faces of the layer, at $z = \pm h$: $W = 0$, $\tau_{xz} = \tau_{yz} = 0$, where W is the projection of the displacement vector on the Oz axis, and τ_{xz}, τ_{yz} are the stress tensor components.

The problem under consideration is reduced to the solution of an integral equation of the first kind by the methods of integral transformation:

$$-\Delta \iint\limits_{\Omega} \gamma(\xi, \eta) \frac{d\xi\, d\eta}{R} + \iint\limits_{\Omega} \gamma(\xi, \eta) K_1\left(\frac{R}{h}\right) d\xi\, d\eta = \frac{2\pi p(x, y)}{\Theta} \tag{1.1}$$

$$(x, y) \in \Omega$$

$$K_1(\alpha) = \frac{1}{h^3} \int_0^\infty [L(u) - 1]\, u^2 J_0(\alpha u)\, du, \quad L(u) = \frac{\operatorname{sh} 2u + 2u}{\operatorname{ch} 2u - 1}$$

$$\gamma(x, y) = W(x, y, 0), \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$R = \sqrt{(x - \xi)^2 + (y - \eta)^2}, \quad \Theta = \frac{E}{2(1 - \nu^2)}$$

where E is Young's modulus, ν is Poisson's ratio, and $J_0(x)$ is the Bessel function of the first kind.

As a result of utilizing the well-known integral representations /1/, the integral equation (1.1) can be converted to the form

*Prikl.Matem.Mekhan., 48, 6, 1030-1038, 1984

$$\iint_{\Omega} \gamma(\xi, \eta) K_2(\xi - x, \eta - y)\, d\xi\, d\eta = \frac{\pi^2 p(x, y)}{\theta}, \quad (x, y) \in \Omega \tag{1.2}$$

$$K_2(\alpha, \beta) = \int_0^\infty \int_0^\infty \sqrt{u^2 + v^2}\, L(h\sqrt{u^2 + v^2}) \cos \alpha u \cos \beta v\, du\, dv$$

We note the following properties of the function $L(u)$:

$$L(u) = 1 + O(ue^{-2u}),\ u \to \infty;\ \lim_{u \to 0} uL(u) = 2 \tag{1.3}$$

which enables us to formulate the following assertion.

Lemma. The function $K_1(\alpha)$ is continuous with all its derivatives for all $0 \leqslant \alpha < \infty$. For $0 \leqslant \alpha < 4$ the function $K_1(\alpha)$ is representable by the absolutely convergent series

$$K_1(\alpha) = \frac{1}{h^3} \sum_{k=0}^{\infty} l_k \alpha^{2k} \tag{1.4}$$

The proof of the lemma results from the properties (1.3) and the representation of the function $J_0(\alpha u)$ as a power series. Direct calculations enable us to obtain ($\zeta(\alpha)$ is the Riemann zeta-function)

$$l_k = \frac{(-1)^k (2k+2)!\,(k+2)\,\zeta(2k+3)}{(k!)^2\, 2^{4k+1}}, \quad k = 0, 1, 2, \ldots$$

We will now examine methods of solving the integral equations obtained.

2. A crack in a layer of large relative thickness. We introduce a dimensionless parameter characterizing the relative thickness of the layer $\lambda = h/a$, where $2a$ is the greatest distance between two points of the contour. The method of constructing the asymptotic solution of integral equations of the form (1.1) for sufficiently large values of λ /2/ is described in detail in /3/.

Let Ω be the elliptical domain $s(x, y) \geqslant 0$; $s(x, y) = 1 - x^2/a^2 - y^2/b^2$; $a \geqslant b$. To be specific we set $p(x, y) = p = \mathrm{const}$. In this case the asymptotic expansion of the solution of the integral equation (1.1) taking (1.4) into account can be represented in the form

$$\gamma(x, y) = \frac{bp}{\theta E(k)} \sqrt{s(x, y)} \left[1 + \frac{c_0 b^2 \pi}{2a^2 E(k) \lambda^3} + O(\lambda^{-5})\right] \tag{2.1}$$

$$c_0 = -\frac{4\zeta(3)}{3\pi}, \quad k = \sqrt{1 - \varepsilon^2}, \quad \varepsilon = \frac{b}{a}$$

where $K(k)$ and $E(k)$ are the complete elliptical integrals of the first and second kinds, respectively.

We will now examine the problem of two symmetric cracks with respect to the axis $x = 0$ that lie in the plane $z = 0$. Let Ω: $\Omega_1 \cup \Omega_2$. In this case, the integral equation (1.1) can be converted to the following form by virtue of the symmetry of the problem

$$-\Delta \iint_{\Omega_1} \gamma(\xi, \eta) \frac{d\xi\, d\eta}{R} + \iint_{\Omega_1} \gamma(\xi, \eta) \left[K_1\left(\frac{R}{h}\right) + \right. \tag{2.2}$$

$$\left. K_3(x + \xi + 2l, \eta - y) + K_4(x + \xi + 2l, \eta - y)\right] d\xi\, d\eta = \frac{2\pi p}{\theta}$$

$$(x, y) \in \Omega_1$$

$$K_3(\alpha, \beta) = -(\alpha^2 + \beta^2)^{-1/2}, \quad K_4(\alpha, \beta) = K_1\left(\frac{\sqrt{\alpha^2 + \beta^2}}{h}\right)$$

Let the domain occupied by the cracks have elliptical shape and be described by the relationships

$$\Omega_1:\ x^2/a^2 + y^2/b^2 \leqslant 1;\ \Omega_2:\ (x + 2l)^2/a^2 + y^2/b^2 \leqslant 1$$

We introduce two dimensionless geometrical parameters

$$\lambda_1 = h/a;\ \lambda_2 = l/a\ (0 < \lambda_1 < \infty;\ 1 < \lambda_2 < \infty)$$

To avoid awkward calculations in constructing the asymptotic expansion of the solution in these two parameters, we introduce the notation $\lambda_1 = t\lambda_2 = t\lambda$. The parameter t changes in such a manner that the parameters λ_2 and λ_1 remain fairly large. Furthermore, by applying the above-mentioned method to solve the integral equation (2.2), we obtain its solution in the form of the asymptotic expansion

$$\gamma(x, y) = \frac{bp}{\theta E(k)} \sqrt{s(x, y)} \left\{1 - \frac{b^2}{a^2 \lambda^3} \left[\frac{g_3}{E(k)} + \frac{g_4 x}{\lambda a \varkappa}\right]\right\} + O(\lambda^{-5}) \tag{2.3}$$

$$\varkappa = [(2k^2 - 1) E(k) + (1 - k^2) K(k)]\, k^{-2}$$

$$g_3 = -\frac{1}{8} - \frac{3\pi}{2t^3} \left[c_0 + \sum_{n=0}^{\infty} \left(\frac{6}{t}\right)^{2n} c_n\right] \tag{2.4}$$

$$g_4 = \frac{3}{2}\left[\frac{1}{8} - \frac{\pi}{t^3}\sum_{n=1}^{\infty}\left(\frac{6}{t}\right)^{2n} nc_n\right]$$

$$c_n = -2l_n/(\pi 3^{n+1}) \quad (n = 0, 1, 2, \ldots)$$

It is convenient to use the integral representations

$$\sum_{n=0}^{\infty}(3\alpha)^{2n} c_n = -\frac{2}{3\pi}\int_0^{\infty}[L(u)-1]\,u^2 J_0(\alpha u)\,du$$

$$\sum_{n=1}^{\infty}(3\alpha)^{2n} nc_n = \frac{\alpha}{3\pi}\int_0^{\infty}[L(u)-1]\,u^3 J_1(\alpha u)\,du$$

for calculations by means of (2.3).

If we pass to the limit as $t \to \infty$ in (2.4), then the solution (2.3) will correspond to the equilibrium problem for an elastic space weakened by two symmetric elliptical cracks.

The solution of the problem of a layer with one elliptical cracks results from (2.3) because of the realization of a simultaneous passage to the limit as $\lambda \to \infty$ and $t \to 0$. The relationships $\lambda_1 = t\lambda$ is here conserved within allowable limits.

3. A crack in a layer of small relative thickness. A "degenerate" solution of the problem on the contact between a stamp and an elastic layer for the case of its small relative thickness was constructed in /1/. By analogous reasoning, we obtain

$$\gamma(x, y) = \frac{h}{2\theta}\left[1 - \frac{1}{45}h^4\Delta^2 + \ldots\right]p(x, y), \quad (x, y) \in \Omega \tag{3.1}$$

Relationship (3.1) yields the "internal" solution of the integral equation (1.1) for small relative layer thicknesses. It will be the more accurate the more deeply the point under consideration is removed from the contour into the interior of the domain Ω along the normal. The accuracy of the constructed degenerate solution is determined by estimates analogous to /1/. Relationship (3.1) becomes meaningless as one approaches the crack contour.

Will now construct a solution of the boundary-layer type in the neighbourhood of the crack contour. Under the condition of axial symmetry of the problem such a solution has been constructed in /4/. An analogous solution of the three-dimensional contact problem is presented in /1/. Realization of similar reasoning in the case under consideration enables the following result to be obtained:

$$\gamma(x, y) = \frac{ph}{2\theta}\left\{\operatorname{erf}\sqrt{Dn(x, y)} + \frac{2A}{\sqrt{\pi}}\exp\left[-BDn(x, y)\right] \times F\left[A\sqrt{Dn(x, y)}\right]\right\} \tag{3.2}$$

$$A = 0.876;\ B = 1.768;\ D = 0.640;\ n(x, y) = r(x, y)/h$$

($r(x, y)$ is the distance between the point and the crack contour). We note that $\gamma(x, y) = ph/(2\theta)$ follows from (3.1) for $p(x, y) = p = \text{const}$. This solution follows from (3.2) when $n \gg 1$.

The reasoning performed and the solution constructed are valid for any simply-connected domain Ω bounded by a smooth contour.

4. Successive approximations. A method of constructing approximate solutions of the integral equations of the first kind under investigation is considered. The solution of the equation is constructed in the form of the product of two functions, one of which is the solution of the equation for one of the limit values of the kernel parameter. An integral equation of the second kind is obtained for the second of the functions mentioned, and its solution can be constructed by the method of successive approximations, say.

Consider the integral equation of the first kind

$$-\Delta\iint_{\Omega}\gamma(\xi, \eta)\frac{d\xi\,d\eta}{R} + \iint_{\Omega}\gamma(\xi, \eta)K(x, \xi, y, \eta)\,d\xi\,d\eta = f(x, y), \quad (x, y) \in \Omega \tag{4.1}$$

The function $K(x, \xi, y, \eta)$ is the regular part of the kernel of the integral equation, $f(x, y)$ is a fairly smooth function, and Ω is a simply-connected domain bounded by a smooth contour.

We construct the solution of (4.1) in the form

$$\gamma(x, y) = \gamma_*(x, y)\,\omega(x, y) \tag{4.2}$$

where $\gamma_*(x, y)$ is the solution of the integral equation (4.1) obtained by well-known methods for the limit value of one of the kernel parameters.

Substituting (4.2) into (4.1), we obtain an integral equation to determine the function $\omega(x, y)$ whose solution can be obtained by the method of successive approximations, where the expression

$$\omega_0(x, y) = M(x, y) f(x, y) \tag{4.3}$$
$$M^{-1}(x, y) = -\Delta \iint_\Omega \gamma_*(\xi, \eta) \frac{d\xi\, d\eta}{R} + \iint_\Omega \gamma_*(\xi, \eta) K(x, \xi, y, \eta)\, d\xi\, d\eta$$

should be taken as the zeroth approximation.

The recursion relations

$$\omega_n(x, y) = M(x, y)\Big\{f(x, y) + \Delta \iint_\Omega \gamma_*(\xi, \eta)\, \omega_{n-1}(\xi, \eta) \frac{d\xi\, d\eta}{R} -$$
$$\omega_{n-1}(x, y)\Delta \iint_\Omega \gamma_*(\xi, \eta) \frac{d\xi\, d\eta}{R} - \iint_\Omega \gamma_*(\xi, \eta) \times$$
$$[\omega_{n-1}(\xi, \eta) - \omega_{n-1}(x, y)]\, K(x, \xi, y, \eta)\, d\xi d\eta\Big\}, \quad (x, y) \in \Omega$$
$$(n = 1, 2, \ldots)$$

are used to calculate subsequent approximations.

We construct the solution of the integral equation (4.1) by the method of successive approximations in the case when the regular part of the kernel is representable by the following asymptotic series

$$K(x, \xi, y, \eta) = \frac{1}{a^3\lambda^3}\left[g_3 + g_4 \frac{x + \xi}{a\lambda}\right] + O(\lambda^{-5}) \tag{4.4}$$

Let the domain of interchange of the boundary conditions Ω be: $s(x, y) \geqslant 0$; $f(x, y) = 2\pi p/\Theta =$ const (g_3, g_4 are arbitrary numbers).

The solution

$$\lambda \to \infty,\ \gamma_*(x, y) = A\sqrt{s(x, y)}\ (A = bp\, [\Theta E(k)]^{-1}) \tag{4.5}$$

corresponds to the limit case.

As a result of calculations using (4.3), taking (4.4) and (4.5) into account we have

$$\omega_0(x, y) = 1 - \frac{g_3 b^2}{3a^2 E(k)\lambda^3} - \frac{g_4 b^2 x}{3a^2 E(k)\lambda^4} + O(\lambda^{-5}) \tag{4.6}$$

and later applying the proposed algorithm, we obtain the following relationship for the successive evaluation of approximations of the function $\omega(x, y)$ desired:

$$\omega_n(x, y) = \omega_{n-1}(x, y) + \omega_0(x, y)\Big\{1 + \frac{A\Theta}{2\pi p}\Big[\Delta\iint_\Omega \sqrt{s(\xi, \eta)}\, \omega_{n-1}(\xi, \eta)\frac{d\xi\, d\eta}{R} - \tag{4.7}$$
$$\iint_\Omega \sqrt{s(\xi, \eta)}\, \omega_{n-1}(\xi, \eta)\, K(x, y, \xi, \eta)\, d\xi\, d\eta\Big]\Big\} \quad (n = 1, 2, \ldots)$$

An asymptotic expansion of the exact solution of the integral equation under investigation in a series of negative powers of the parameter λ was constructed above. For $a \gg b$ the following relationship corresponds to it:

$$\omega^* = 1 - \frac{g_3 b^2}{3a^2 E(k)\lambda^3} - \frac{g_4 b^2}{3a^2\lambda^4}\frac{x}{a\varkappa} + O(\lambda^{-5}) \tag{4.8}$$

It is seen from a comparison of (4.6) and (4.8) that the zeroth approximation for the function $\omega(x, y)$ already yields exact values of terms in the expansion $\omega_n(x, y)$ for λ^{-3} and lower orders of smallness.

Realization of the expansion (4.4) and (4.6) in (4.7) and evaluation of the corresponding integrals enable us to obtain recursion relations to calculate terms in λ^{-4} and higher order.

If the following notation is used

$$\omega_n(x, y) = 1 - \alpha_n \frac{g_3 b^2}{3a^2 E(k)\lambda^3} - \beta_n \frac{g_4 b^2 x}{3a^3\lambda^4} + O(\lambda^{-5})$$

then

$$\alpha_n = \alpha_0 = 1,\ \beta_n = \beta_0 + [1 - \varkappa/E(k)]\, \beta_{n-1},\ \beta_0 = 1/E(k) \tag{4.9}$$

The appropriate coefficients of the expansion of the exact solution are calculated from the formulas

$$\alpha^* = 1,\ \beta^* = 1/\varkappa \tag{4.10}$$

It can be seen that $-1/2 \leqslant 1 - \varkappa/E(k) \leqslant 0$. This indicates that the sequence of values of β_n will converge, and its limit will be β^*.

In particular, for $k^2 = 0.6$ direct calculations by means of (4.9) and (4.10) yield

$$\lim_{n\to\infty} \beta_n = \beta^* = 0.5776$$

Therefore, the expansion of the solution obtained by successive approximations in the parameter λ^{-1} will converge entirely to the asymptotic expansion of the exact solution of the problem.

The limits of applicability of the solution obtained by such means will be determined by the radius of convergence of the expansion (4.4). In practice, the zeroth approximation in the form (4.5) can be used as the approximate solution of the initial equation for $4 \leqslant \lambda < \infty$. If exact representations are taken for the kernels, then the domain in which the zeroth approximation can be taken as an approximate solution with an accuracy sufficient for practice is boradened substantially.

Calculations using (4.3) in the problem being considered here enable us to obtain

$$\omega_0(x, y) = \left[1 + \frac{ab}{2\pi A E(k)} \iint_\Omega \sqrt{s(\xi, \eta)}\, K(x, \xi, y, \eta)\, d\xi\, d\eta\right]^{-1} \tag{4.11}$$

5. A variational method. The methods examined above enable a solution of the integral equation under investigation to be constructed in a form that is simple in structure. In order to set the limits within which the solutions obtained yield sufficient accuracy for practical purposes, a variational method that yields guaranteed accuracy is also applied. The solution of the integral equation in the form (1.2) will here be sought in the case of an elliptical domain Ω in the following form:

$$\gamma(x, y) = A\sqrt{s(x, y)} \sum_{m=0}^{M} \sum_{n=0}^{N} A_{mn} \cos\left(m\pi \frac{x}{a}\right) \cos\left(n\pi \frac{y}{b}\right) \tag{5.1}$$

We determine the coefficients A_{mn} by Ritz's method from the condition for a minimum of the functional

$$J(\gamma) = \iint_\Omega \gamma(x, y) \iint_\Omega \gamma(\xi, \eta) K_2(\xi - x, \eta - y)\, d\xi\, d\eta\, dx\, dy - 2\frac{\pi^2}{\theta} \iint_\Omega \gamma(x, y)\, p(x, y)\, dx\, dy \tag{5.2}$$

We note that an analogous approach to the solution of integral equations of a somewhat different kind is used in /5/, where power-law functions were selected as the coordinate functions.

Substituting (5.1) into (5.2) and writing down the condition for a minimum of the functional $J(\gamma)$, we obtain a system of linear algebraic equations for determining the coefficients A_{mn}

$$\sum_{i=0}^{M} \sum_{j=0}^{N} P_{ijmn} A_{ij} = \pi E(k) F(\pi m, \pi n) \tag{5.3}$$

$$(m = 0, 1, 2, \ldots, M;\ n = 0, 1, 2, \ldots, N)$$

$$F(\alpha, \beta) = \sin H/H^3 - \cos H/H^2,\quad H = \sqrt{\alpha^2 + \beta^2}$$

$$P_{ijmn} = \frac{1}{8} \int_0^\infty \int_0^\infty S_{ijmn}(\alpha, \beta)\, L(\lambda\sqrt{(\alpha\varepsilon)^2 + \beta^2})\, d\alpha\, d\beta \tag{5.4}$$

$$S_{ijmn}(\alpha, \beta) = [F(\alpha - \pi m, \beta - \pi n) + F(\alpha - \pi m, \beta + \pi n) + F(\alpha + \pi m, \beta - \pi n) + F(\alpha + \pi m, \beta + \pi n)]\,[F(\alpha - \pi i, \beta - \pi j) + F(\alpha - \pi i, \beta + \pi j) + F(\alpha + \pi i, \beta - \pi j) + F(\alpha + \pi i, \beta + \pi j)]\sqrt{(\alpha\varepsilon)^2 + \beta^2}$$

It is seen from (5.4) that $P_{ijmn} = P_{mnij}$.

It is convenient to represent P_{ijmn} in the form $P_{ijmn} = P_{ijmn}^\infty + P_{ijmn}^*$ for calculations of the coefficients, where P_{ijmn}^∞ corresponds to the limit case $\lambda \to \infty$. The integrand in P_{ijmn}^* decreases exponentially for large values of the argument by virtue of the properties (1.3) of the function $L(u)$.

The variational method of solving the integral equation (1.2) can be applied in an analogous form even in the case when the crack occupies a rectangular domain.

Let the domain of integration Ω in the functional (5.2) be $\{|x| \leqslant a, y \leqslant b\}$. As before, in the neighbourhood of points of crack contour smoothness the desired solution should have the asymptotic form

$$\gamma(x, y) = \Gamma_0(x, y)\, \rho_*^{1/2} \tag{5.5}$$

where ρ is the distance between the point with coordinates (x, y) and the contour. As established in /6/, the asymptotic form of the behaviour of the function $\gamma(x, y)$ near an angular point of the crack contour is different, and in particular, has the following form in the case of a right angle:

$$\gamma(x, y) = B(\alpha)(r/a)^{0,816} \tag{5.6}$$

where r, α are polar coordinates with pole at the angular point of the crack contour.

As in the problem of an elliptical crack, we select a system of coordinate functions that explicitly take account of the asymptotic form (5.5). We seek the solution of the problem in the case when $p(x, y) = p = \text{const}$ in the form

$$\gamma(x, y) = a\frac{p}{\theta}\sqrt{\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)}\sum_{m=0}^{M}\sum_{n=0}^{N} A_{mn}\cos\left(m\pi\frac{x}{a}\right)\cos\left(n\pi\frac{y}{b}\right) \tag{5.7}$$

An analogous solution of the problem on the equilibrium of an elastic space weakened by a rectangular crack* (Gol'dshtein R.V., Entov, V.M. and Zazovskii A.F. Solution of mixed boundary value problems by a direct variational method. Preprint No.78, Inst. Problem Mekhaniki Akad. Nauk SSSR, Moscow, 1976, 54p.) can be obtained from (5.7) by passing to the limit as $\lambda \to \infty$.

Like (5.3), the coefficients of the expansion are determined from the following system of linear algebraic equations:

$$\sum_{i=0}^{M}\sum_{j=0}^{N} P_{ijmn}A_{ij} = 2\varepsilon F(\pi m, \pi n), \quad F(\alpha, \beta) = \frac{J_1(\alpha)J_1(\beta)}{\alpha\beta} \tag{5.8}$$
$$(m = 0, 1, 2, \ldots, M;\ n = 0, 1, 2, \ldots, N)$$

(the meaning of all the notation except $F(\alpha, \beta)$ is retained from (5.4)).

6. Analysis and comparison of the numerical results. It is convenient to introduce the quantity $N = K_I/K_{I\infty}$ in discussing the results of an investigation of the problem of an elliptical crack in an elastic layer, where K_I is the normal stress intensity coefficient in the problem under consideration and $K_{I\infty}$ corresponds to the limit case as $\lambda \to \infty$. It can be seen that

$$N = \lim_{\rho\to 0}[\gamma(x, y)/\gamma_\infty(x, y)] \tag{6.1}$$

Thus, we obtain for the solution of the problem by the method of "large λ" (Γ is the contour of the domain Ω)

$$N = 1 - \frac{b^2}{3a^2\lambda^3}\left[\frac{g_3}{E(k)} + \frac{g_4 x}{\lambda a \varkappa}\right] + O(\lambda^{-5}), \quad (x, y) \in \Gamma \tag{6.2}$$

We obtain the value of N at the vertices of the elliptical domain from (3.2) by solving the problem by the method of "small λ"

$$N|_{(a,0)} = \frac{E(k)}{b}\sqrt{\frac{ah}{\pi}}, \quad N|_{(0,b)} = \frac{E(k)}{b}\sqrt{\frac{bh}{\pi}} \tag{6.3}$$

We note that because of the passage to the limit as $a/b \to \infty$ a corresponding results is obtained for a strip crack of width $2b$ in a layer /4/ from (6.3)

$$N = \sqrt{h/(\pi b)}$$

The method described in Sect.4 enables the following result to be obtained

$$N = \omega_0(x, y)|_\Gamma \tag{6.4}$$

Finally, as a result of solving the problem by a variational method we can write the expression for the quantity being investigated in the form

$$N = \sum_{m=0}^{M}\sum_{n=0}^{N} A_{mn}\cos\left(m\pi\frac{x}{a}\right)\cos\left(n\pi\frac{y}{b}\right)\Big|_\Gamma \tag{6.5}$$

It is convenient to set $x = a\cos\varphi$, $y = b\sin\varphi$ in (6.2)−(6.5) as the point (x, y) traverses the crack contour, where the angle φ is measured from the positive direction of the Ox axis.

Results of a calculation of the quantity N by means of (6.2)−(6.5) are shown in Fig.1 for $b/a = 0.5$; $\varphi = 0$ and $\varphi = \pi/2$ as a function of the relative layer thickness λ.

The results obtained indicate that the solution of the problem considered here for an elliptical crack in an elastic layer by the method of large λ can be used with sufficient accuract for practical purposes in the range $\lambda \geqslant 1.5$ (the error does not exceed 2%). The method described in Sect.4 turns out to be effective for $\lambda \geqslant 0.5$. The solution of the problem by the method of small λ yields sufficient accuracy only for very small relative layer thicknesses ($\lambda \leqslant 0.5$). Investigation of the problem by a variational method yields a practically exact solution in all the cases considered ($\lambda \geqslant 0.25$), and this method is used in the

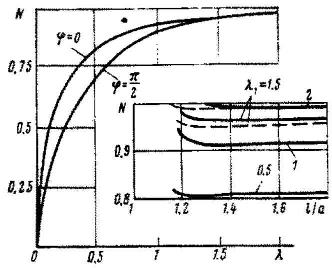

Fig.1

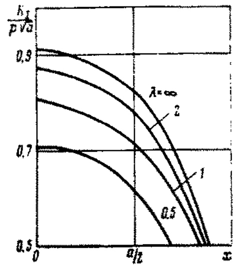

Fig.2

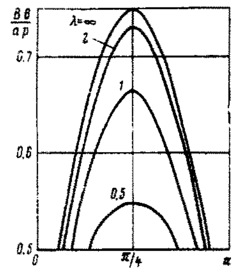

Fig.3

problem mentioned only to esimate the accuracy of the asymptotic solutions obtained.

Certain results of an investigation of the problem of the equilibrium of an elastic layer weakened by two symmetric elliptical cracks are displaced in the insert to Fig.1. The calculations are performed at the point of the crack contour Ω_1 where their mutual influence is most perceptible $(\varphi = \pi)$. The results are given for $\lambda_1 = 2, 1.5, 1, 0.5$. The results of an investigation of the problem by the method described in Sect.4 are represented by the solid lines. Here the results of solving the problem by the method of large λ are represented by dashes for comparison, for the case of a fairly large relative layer thickness $(\lambda_1 = 2, 1.5)$. The calculations showed that the disagreement between the results of calculations by the method of large λ and by the method described above do not exceed 3% in the following ranges of the geometric parameters of the problem $h/a \geqslant 1.5$, $l/a \geqslant 1.5$, $b/a \leqslant 1.5$.

We will now analyse the numerical values of the normal stress intensity coefficient K_I for traversal around the contour of a rectangular crack. In this case it turns out to be more convenient to examine its absolute values since there is not exact solution to this problem in the limit case as $\lambda \to \infty$.

The normal stress intensity coefficients along each of the sides of the rectangle are evaluated from the formulas

$$K_I^a = p\sqrt{\frac{2a^2}{b}}\sqrt{1-\frac{x^2}{a^2}}\sum_{m=0}^{M}\sum_{n=0}^{N} A_{mn}\cos\left(m\pi\frac{x}{a}\right) \tag{6.6}$$

$$K_I^b = p\sqrt{2a}\sqrt{1-\frac{y^2}{b^2}}\sum_{m=0}^{M}\sum_{n=0}^{N} A_{mn}\cos\left(n\pi\frac{y}{b}\right)$$

Graphs of the change in the quantity $K_I/(p\sqrt{a})$ along the larger side $(b/a = 0.5)$ are displayed in Fig.2 for different values of λ.

We note that the problems considered in the paper for the conditions on mentioned the layer faces correspond physically also to the equilibrium of an elastic space weakened by a chain of cracks. The cracks are in parallel planes at a distance of $2h$.

The solution $\gamma(x, y)$ in the neighbourhood of angular points of the crack was constructed as follows. Along rays emerging from the angular point at a different angle α values of the ratio $B(\alpha) = \gamma(x, y)/(r/a)^{0.816}$ were calculated. Direct calculations enable us to find that there is an interval of variation in r/a in which the mentioned ratio remains constant along each ray (the deviations do not exceed 4%) and varies only when passing from one ray to another. This regularity was found earlier for the limit case of the problem being considered here, corresponding to $\lambda \to \infty$ (see the footnote on p. 762). The location and size of this zone specify the relative thickness of the layer λ and other geometric parameters. Therefore, by determining the quantity $B(\alpha)$ for each value of α it turns out to be possible to construct the solution $\gamma(x, y)$ directly to the apex of the crack angle by the asymptotic formula (5.6).

Graphs of the function $B(\alpha)$ are presented in Fig.3 for different relative layer thickness for $b/a = 0.5$. Note that although the domain occupied by the crack is not symmetric relative to the angle bisector, this symmetry is observed for the function $\gamma(x, y)$ in the neighbourhood of the angular point of the contour.

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Translated by M.D.F.

PMM U.S.S.R., Vol.48, No.6, pp.764-767, 1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

A CLASS OF MOTIONS OF A TOP IN THE GORYACHEV-CHAPLYGIN CASE*

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A solution of the Euler-Poisson equations is studied for the Goryachev-Chaplygin case /1/ under the condition when the ultraelliptic integrals degenerate to elliptic /2/. A solution is constructed for a class of motions in which both quantities, u and v, brought in by Chaplygin, vary with time, but one of them tends asymptotically to a constant when the time increases without limit. The dependence of the Euler-Poisson variables on time is expressed in terms of elliptic functions and an elliptic integral of the third kind. Fairly simple approximate formulas are given for determining all six variables sought.

1. Equations of motion. We will use the Goryachev-Chaplygin conditions and take the Euler-Poisson equations in traditional form /1/ (a dot denotes differentiation with respect to time)

$$4p^{\cdot} = 3qr,\ 4q^{\cdot} = -3rp - a\gamma'',\ r^{\cdot} = a\gamma' \qquad (1.1)$$
$$\gamma^{\cdot} = r\gamma' - q\gamma'',\ \gamma'^{\cdot} = p\gamma'' - r\gamma,\ \gamma''^{\cdot} = q\gamma - p\gamma'$$

If the area constant is zero, the above system admits of four algebraic integrals

$$4(p^2+q^2) + r^2 - 2a\gamma = k,\ \gamma^2 + \gamma'^2 + \gamma''^2 = 1 \qquad (1.2)$$
$$4(p\gamma + q\gamma') + r\gamma'' = 0,\ r(p^2+q^2) + ap\gamma'' = g$$

We introduce two auxiliary variables u, v so that

$$u + v = r,\ uv = -4(p^2+q^2) \qquad (1.3)$$

The following differential equations describe how these quantities vary with time:

$$2(u-v)u^{\cdot} = \sqrt{F(u)},\quad 2(v-u)v^{\cdot} = \sqrt{F(v)} \qquad (1.4)$$
$$F(u) = f_1(u) f_2(u)$$
$$f_1(u) = -u^3 + (k+2a)u + 4g,\ f_2(u) = u^3 + (2a-k)u - 4g$$

System (1.4) can be written in terms of total differentials thus

$$\frac{du}{\sqrt{F(u)}} + \frac{dv}{\sqrt{F(v)}} = 0,\quad \frac{2u\,du}{\sqrt{F(u)}} + \frac{2v\,dv}{\sqrt{F(v)}} = dt \qquad (1.5)$$

Let us write $(k-2a)^3 + 27\cdot 4g^2 = 0$, or in parametric form (b is an auxiliary constant)

$$k - 2a = 3b^2,\ 2g = -b^3 \qquad (1.6)$$

Then $f_2(u) = (u-b)^2(u+2b)$, $f_1(u) = -(u-\alpha_1)(u-\alpha_2)(u-\alpha_3)$ where all three roots $\alpha_1, \alpha_2, \alpha_3$ are real and $\alpha_1 < \alpha_2 < 0 < \alpha_3$, $-2b < \alpha_3$. It follows that the polynomial $F(u)$ has a multiple root

$$F(u) = (u-b)^2 R(u),\ R(u) = (u+2b) f_1(u) \qquad (1.7)$$

Let us describe the type of set in which the variables u, v vary. We shall assume that $g \neq 0$, since when $g = 0$ the solution is known /3, 4/. Then $p^2 + q^2 \neq 0$ and hence by virtue of (1.3), $uv \neq 0$. We can assume without loss of generality that $u > 0, v < 0, b < 0$. Bearing this in mind, we obtain

$$0 < -2b \leqslant u \leqslant \alpha_3 \qquad (1.8)$$

Thus the quantity u varies on the interval (1.8). The set of variations is more complicated for the second variable v. Depending on the initial data, three versions are possible 1) $\alpha_1 \leqslant v < b < 0$, 2) $b < v \leqslant \alpha_2$, 3) $v = b = \text{const}$. The last version is relatively simple and

*Prikl.Matem.Mekhan., 48, 6, 1039-1042, 1984